Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives

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Abstract

The classical fields with fractional derivatives are investigated by using the fractional Lagrangian formulation.The fractional Euler-Lagrange equations were obtained and two examples were studied.

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1 Introduction

Fractional derivatives [1, 2, 3], have played a significant role in physics, mathematics and engineering during the last decade [4, 5, 6, 7, 8]. Fractional calculus found many interesting applications in recent studies of scaling phenomena [9, 10, 11] or in classical mechanics [12, 13, 14, 15, 16].

Riewe has used the fractional calculus to develop a formalism which can be used for both conservative and non conservative systems [12, 13]. Although many laws of nature can be obtained using certain functionals and the theory of calculus of variations, not all laws can be obtained by using this way. For example, almost all systems contain internal damping, yet traditional energy based approach cannot be used to obtain equations describing the behavior of a non conservative system [12, 13]. Using the fractional approach, one can obtain the Euler-Lagrange and the Hamiltonian equations of motion for the nonconservative systems. The simple solutions of the fractional Dirac equation of order $\frac{2}{3}$ were investigated recently [17, 18].Even more recently the fractional variational principle in macroscopic picture was discussed in [19].

Recently, an extension of the simplest fractional problem and the fractional variational problem of Lagrange was obtained [15, 16]. A natural generalization of Agrawal's approach, was to apply the fractional calculus to constrained systems [20] and to obtain both the fractional Euler-Lagrange equations [21] and the fractional Hamiltonian formulation of constrained systems [22, 23].

The fractional Lagrangian is non-local, therefore we should take care of this property in handling with its corresponding Hamiltonian. An interesting proposal for the Hamiltonian formalism corresponding to the non-local Lagrangian systems was considered in [24]. The physical degrees of freedom of non-local theories was investigated very recently in [25]. Besides,the Hamiltonian formalism for non-local field theories [26] in d space-time dimensions was developed recently in [27].

For these reasons the fractional variational problems for fields are interesting to be investigated.

The aim of this paper is to obtain the Euler-Lagrange equations for the classical fields with Riemann-Liouville fractional derivatives.

The present paper is organized as follows. In section 2 the fractional Euler-Lagrange equations for fields are obtained. In section 3 the fractional Klein-Gordon equations and Dirac's equation of order $\frac{2}{3}$ are obtained.Finally,

the section 4 is dedicated to our conclusions.

2 Lagrangian formulation of field systems with fractional derivatives

2.1 Riemann-Liouville partial fractional derivative

Let us consider a function f depending on n variables, $x_1, x_2, \cdots x_n$. A partial left Riemann-Liouville fractional derivative of order α_k, $0 < \alpha_k < 1$, in the k-th variable is defined as [1,2]

$$(\mathbf{D}_{a_k+}^{\alpha_k} f)(x) = \frac{1}{\Gamma(1 - \alpha_k)} \frac{\partial}{\partial x_k} \int_{a_k}^{x_k} \frac{f(x_1, \cdots, x_{k-1}, u, x_{k+1}, \cdots, x_n)}{(x_k - u)^{\alpha_k}} du \quad (1)$$

and a partial right Riemann-Liouville fractional derivative of order α_k has the form

$$(\mathbf{D}_{a_k-}^{\alpha_k} f)(x) = -\frac{1}{\Gamma(1 - \alpha_k)} \frac{\partial}{\partial x_k} \int_{x_k}^{a_k} \frac{f(x_1, \cdots, x_{k-1}, u, x_{k+1}, \cdots, x_n)}{(-x_k + u)^{\alpha_k}} du \quad (2)$$

If the function f is differentiable we obtain

$$
\begin{aligned}
(\mathbf{D}_{a_k+}^{\alpha_k} f)(x) &= \frac{1}{\Gamma(1 - \alpha_k)} [\frac{f(x_1, \cdots, x_{k-1}, a_k, x_{k+1}, \cdots, x_n)}{(x_k - a_k)^{\alpha_k}} \\
&+ \int_{a_k}^{x_k} \frac{\frac{\partial f}{\partial u}(x_1, \cdots, x_{k-1}, u, x_{k+1}, \cdots, x_n)}{(x_k - u)^{\alpha_k}}] du.
\end{aligned} \quad (3)
$$

We notice that the last term of the equation (3) represents the Caputo derivative [28]. This derivative has the advantage that certain initial conditions are easier to interpret.

2.2 Fractional classical fields

A covariant form of the action would involve a Lagrangian density $\mathcal{L}$ via $S = \int \mathcal{L} d^4 x = \int \mathcal{L} d^3 x dt$ where $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ and with $L = \int \mathcal{L} d^3 x$. The corresponding covariant Euler-Lagrange equations are

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$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0, \tag{4}$$

where ϕ is the field variable .

Now we shall investigate the fractional generalization of the above Lagrangian density.

Let us consider the action function of the form

$$S = \int \mathcal{L}\left(\phi(x), (\mathbf{D}^{\alpha_k}_{a_k-})\phi(x), (\mathbf{D}^{\alpha_k}_{a_k+})\phi(x), x\right) d^3x dt. \tag{5}$$

Here $0 < \alpha_k \leq 1$ and a_k correspond to x_1, x_2, x_3 and t respectively. For each partial derivative we may have a specific order α_k and a given limit a_k.

In this paper the integration limits are $-\infty$ and ∞ respectively.Under this conditions $\mathbf{D}^{\alpha_k}_{a_k+}$ would become $\mathbf{D}^{\alpha_k}_{-\infty+}$ and $\mathbf{D}^{\alpha_k}_{a_k-}$ would become $\mathbf{D}^{\alpha_k}_{\infty-}$.

Let us consider the ϵ finite variation of the functional $S(\phi)$, that we write with explicit dependence from the fields and their fractional derivatives

$$
\begin{aligned}
\Delta_\epsilon S(\phi) &= \int [\mathcal{L}(x, \phi + \epsilon\delta\phi, (\mathbf{D}^{\alpha_k}_{\infty-})\phi(x) + \epsilon(\mathbf{D}^{\alpha_k}_{\infty-})\delta\phi, (\mathbf{D}^{\alpha_k}_{-\infty+})\phi(x) \\
&+ \epsilon(\mathbf{D}^{\alpha_k}_{-\infty+})\delta\phi) - \mathcal{L}(x, \phi, (\mathbf{D}^{\alpha_k}_{\infty-})\phi(x), (\mathbf{D}^{\alpha_k}_{-\infty+})\phi(x))]d^3x dt. \tag{6}
\end{aligned}
$$

We will develop the first term in square brackets, which is a function on ϵ as a Taylor series in ϵ, stopping at the first order. Therefore from (6) we obtain

$$
\begin{aligned}
\Delta_\epsilon S(\phi) &= \int [\mathcal{L}(x, \phi, (\mathbf{D}^{\alpha_k}_{\infty-})\phi(x), (\mathbf{D}^{\alpha_k}_{\infty+})\phi(x)) + (\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi)\epsilon \\
&+ \sum_k^{1,4} \frac{\partial \mathcal{L}}{\partial(\mathbf{D}^{\alpha_k}_{\infty-}\phi)}\delta(\mathbf{D}^{\alpha_k}_{\infty-}\phi)\epsilon + \sum_k^{1,4} \frac{\partial \mathcal{L}}{\partial(\mathbf{D}^{\alpha_k}_{-\infty+}\phi)}\delta(\mathbf{D}^{\alpha_k}_{-\infty+}\phi)\epsilon + O(\epsilon) \\
&- \mathcal{L}(x, \phi, (\mathbf{D}^{\alpha_k}_{\infty-})\phi(x), (\mathbf{D}^{\alpha_k}_{-\infty+})\phi(x))]d^3x dt \tag{7}
\end{aligned}
$$

Taking into account (7) the form of (6) becomes

$$
\begin{aligned}
\Delta_\epsilon S(\phi) &= \epsilon \int [\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \sum_k^{1,4} \frac{\partial \mathcal{L}}{\partial(\mathbf{D}^{\alpha_k}_{\infty-}\phi)}(\mathbf{D}^{\alpha_k}_{\infty-}\delta\phi) \\
&+ \sum_k^{1,4} \frac{\partial \mathcal{L}}{\partial(\mathbf{D}^{\alpha_k}_{-\infty+}\phi)}(\mathbf{D}^{\alpha_k}_{-\infty+}\delta\phi) + O(\epsilon)]d^3x dt. \tag{8}
\end{aligned}
$$

4

We now perform a fractional integration by parts of the second term in (8) by using the formula [1, 29]

$$\int_{-\infty}^{\infty} f(x)(\mathbf{D}_{-\infty+}^{\alpha_k}g)(x)dx = \int_{-\infty}^{\infty} g(x)(\mathbf{D}_{\infty-}^{\alpha_k}f)(x)dx. \tag{9}$$

Therefore we obtain

$$\begin{aligned}
\Delta_\epsilon S(\phi) &= \epsilon \int [\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \sum_{k}^{1,4}\{(\mathbf{D}_{-\infty+}^{\alpha_k})\frac{\partial \mathcal{L}}{\partial(\mathbf{D}_{\infty-}^{\alpha_k})\phi}\}\delta\phi \\
&+ \sum_{k}^{1,4}\{(\mathbf{D}_{\infty-}^{\alpha_k})\frac{\partial \mathcal{L}}{\partial(\mathbf{D}_{-\infty+}^{\alpha_k})\phi}\}\delta\phi]d^3xdt + \int O(\epsilon)d^3xdt. \tag{10}
\end{aligned}$$

After taking the limit $lim_{\epsilon \longrightarrow 0}\frac{\Delta_\epsilon S(\phi)}{\epsilon}$ we obtain the fractional Euler-Lagrange equations as follows

$$\frac{\partial \mathcal{L}}{\partial \phi} + \sum_{k}^{1,4}\{(\mathbf{D}_{-\infty+}^{\alpha_k})\frac{\partial \mathcal{L}}{\partial(\mathbf{D}_{\infty-}^{\alpha_k})\phi} + (\mathbf{D}_{\infty-}^{\alpha_k})\frac{\partial \mathcal{L}}{\partial(\mathbf{D}_{-\infty+}^{\alpha_k})\phi}\} = 0. \tag{11}$$

It is worth commenting that for $\alpha_k \to 1$, the equations (11) are the usual Euler-Lagrange equations for classical fields.

3   Examples

In this section we shall analyze two systems. The first one is the Klein-Gordon field and the second one is the Dirac field.

3.1   Fractional Klein-Gordon equation

As a first example let us consider the following Lagrangian density as

$$\mathcal{L}_{\mathcal{KG}} = \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2] \tag{12}$$

or

$$\mathcal{L}_{\mathcal{KG}} = \frac{1}{2}[(\dot{\phi})^2 - (\nabla\phi)^2 - m^2\phi^2]. \tag{13}$$

The fractional generalization of (13) is given by

$$\mathcal{L}_{\mathcal{FKG}} = \frac{1}{2}[(\mathbf{D}^{\alpha_t}_{-\infty+}\phi)^2 - (\mathbf{D}^{\alpha_x}_{-\infty+}\phi)^2 - (\mathbf{D}^{\alpha_y}_{-\infty+}\phi)^2 - (\mathbf{D}^{\alpha_z}_{-\infty+}\phi)^2 - m^2\phi^2]. \quad (14)$$

By using (11), the fractional Euler-Lagrange equations are obtain as follows

$$\mathbf{D}^{\alpha_t}_{\infty-}(\mathbf{D}^{\alpha_t}_{-\infty+})\phi - \mathbf{D}^{\alpha_x}_{\infty-}(\mathbf{D}^{\alpha_x}_{-\infty+})\phi - \mathbf{D}^{\alpha_y}_{\infty-}(\mathbf{D}^{\alpha_y}_{-\infty+})\phi - \mathbf{D}^{\alpha_z}_{\infty-}(\mathbf{D}^{\alpha_z}_{-\infty+})\phi - m^2\phi = 0. \quad (15)$$

For all $\alpha_k \to 1$ we obtain the usual Klein-Gordon equation.

3.2 Fractional Dirac equation

The Dirac Lagrangian is given by

$$\mathcal{L}_{\mathcal{D}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \quad (16)$$

In [17, 18] a detailed analysis of the solutions of the fractional Dirac equation of $\frac{2}{3}$ was done.

We propose the Lagrangian density for Dirac field of order $\frac{2}{3}$ as follows

$$\mathcal{L}_{\mathcal{FD}} = \bar{\psi}\left(\gamma^\alpha\mathbf{D}^{2/3}_{\alpha-}\psi(x) + (m)^{2/3}\right)\psi(x). \quad (17)$$

Taking into account (11) and (17) the corresponding Dirac equation becomes

$$\gamma^\alpha\mathbf{D}^{2/3}_{\alpha+}\Psi(x) + (m)^{2/3}\Psi(x) = 0. \quad (18)$$

We would like to stress that in (17) and (18) the expressions of $\mathbf{D}^{2/3}_{\alpha-}$ and $\mathbf{D}^{2/3}_{\alpha+}$ have the same meaning as it was introduced in the previous paragraph but for the sake of simplicity we kept the compact form for summations $\gamma^\alpha\mathbf{D}^{2/3}_{\alpha-}$ and $\gamma^\alpha\mathbf{D}^{2/3}_{\alpha+}$ respectively.

This result is the same as the fractional Dirac equation obtained in references [17, 18].

4 Conclusion

The fractional Lagrangian is not unique due to the fact that we have several choices to replace the fractional derivatives in the usual one.This property of the fractional Lagrangian is an advantage of this theory. For a specific problem we have several options to obtain, under the limit process, the usual Euler-Lagrange equations for fields.

In this paper we have extended the derivation of the usual Euler-Lagrange equations of motion for classical field to the case the Lagrangian contains fractional derivatives of fields. This method has been applied with the variational principle to obtain the corresponding fractional Euler-Lagrange equations. The fractional Klein-Gordon equations were obtained by using the fractional variational principle. This approach allows to obtain the Dirac equation with fractional derivatives of order 2/3 recently obtained in [17, 18].

Acknowledgments

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